

The phase space view of $f(R)$ gravity

José C. C. de Souza^{1,2} and Valerio Faraoni²

¹ Departamento de Física Matemática, Instituto de Física, Universidade de São Paulo, C.P. 66318, 05315-970, São Paulo, SP, Brazil.

² Physics Department, Bishop's University, Sherbrooke, Québec, Canada J1M 0C8

E-mail: jsouza@ubishops.ca, vfaraoni@ubishops.ca

Abstract. We study the geometry of the phase space of spatially flat Friedmann-Lemaître-Robertson-Walker models in $f(R)$ gravity, for a general form of the function $f(R)$. The equilibrium points (de Sitter spaces) and their stability are discussed, and a comparison is made with the phase space of the equivalent scalar-tensor theory. New effective Lagrangians and Hamiltonians are also presented.

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1. Introduction

The observation of type Ia supernovae [1] tells us that the universe is accelerating, while cosmic microwave background experiments [2] point to a spatially flat universe. Within the context of general relativity, the acceleration is explained by a form of dark energy which accounts for 70% of the cosmic energy content with exotic equation of state $P \approx -\rho$. It is even possible that the effective equation of state of this dark energy component has $P < -\rho$ (phantom energy) [3]. As an alternative to postulating this strange form of unseen energy, various modifications of gravity have been proposed. Among them probably the most popular is the so-called $f(R)$ gravity [4], described by the action

$$S = \int d^4x \sqrt{-g} \frac{f(R)}{2} + S^{(matter)}, \quad (1.1)$$

where R is the Ricci curvature, $f(R)$ is a non-linear function of R , g is the determinant of the metric tensor g_{ab} , and $8\pi G = 1$ (G being Newton's constant) in our notations. Other notations follow Ref. [5]. $f(R)$ gravity comes in three versions: the so-called *metric formalism*, in which the action (1.1) is varied with respect to g^{ab} and the field equations are of fourth order; the *Palatini formalism* [6], in which the action (1.1) is varied with respect to both the metric and the connection, yielding second order equations, and in which the matter part of the action $S^{(matter)}$ is independent of the connection; and the so-called *metric-affine gravity* version in which also $S^{(matter)}$ depends explicitly on it [7]. The physical motivation for all these versions of modified gravity lies in the desire to explain the cosmic acceleration without dark energy (and, especially, to get away from the embarrassing notion of phantom energy).

Here we restrict the analysis to the metric formalism, in which the field equations are

$$f'(R)R_{ab} - \frac{f(R)}{2}g_{ab} = \nabla_a \nabla_b f' - g_{ab} \square f' + T_{ab} \quad (1.2)$$

(a prime denotes differentiation with respect to R).

While several models of metric $f(R)$ cosmology are found in the literature, most works consider specific forms of $f(R)$ which are not motivated by fundamental reasons, and are usually chosen on the basis of simplicity or ease of calculation. It is more interesting not to make assumptions on the form of $f(R)$ and try instead to understand this class of theories in as general a way as possible, without choosing the form of $f(R)$. In a previous paper we have shown that theories with $f'' < 0$ are not viable due to instabilities in the Ricci scalar [8, 9]. Problems with the weak field limit have also been pointed out for many choices of $f(R)$ [10]. Other issues to be studied include the correct sequence of cosmological eras (inflation, radiation era, matter era, present accelerated era) and whether smooth transitions between them are possible [11]; the presence of ghosts and instabilities [12]; and the well-posedness of the Cauchy problem [8].

Here we provide a description of the phase space for general metric $f(R)$ cosmology, i.e., without specifying the form of the function $f(R)$. We focus on the geometry of the

phase space, the existence of equilibrium points and their stability, and the phase space picture of the equivalent scalar-tensor theory.

Further motivation for our study comes from scenarios of inflation in the early universe employing quadratic corrections to the Einstein-Hilbert Lagrangian, i.e., $f(R) = R + aR^2$ [13]. (The phase space of such theories has already received attention in the literature [14].) Quadratic corrections are motivated by attempts to renormalize Einstein's theory [15], and are included as special cases in the following discussion.

It is well known that $f(R)$ gravity can be mapped into the Einstein conformal frame in which the theory is equivalent to a scalar field minimally coupled to the Ricci curvature ([16]; see also [17, 18]). Although this version is completely equivalent to “Jordan frame” $f(R)$ gravity, at least at the classical level, there have been much confusion and misunderstanding about the mapping into the Einstein frame (see, e.g., the recent discussion of [19]) and we prefer to proceed directly without using this conformal mapping.

The plan of this paper is as follows. In Sec. 2, the geometry of the phase space of spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology is studied. In Sec. 3, the phase space view is compared with that of the equivalent scalar-tensor theory, and effective Lagrangians and Hamiltonians are presented for both cases. Sec. 4 contains a discussion and the conclusions.

2. The phase space of $f(R)$ gravity

In this section, we study the geometry of the phase space of $f(R)$ gravity. Since we study regimes dominated by corrections to Einstein gravity (for example, the late-time acceleration of the universe, or the early epoch of inflation in Starobinsky-like scenarios [13]), we omit the matter part of the action (1.1). Motivated by the recent cosmological observations, we restrict the analysis to the spatially flat FLRW line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \quad (2.1)$$

in comoving coordinates (t, x, y, z) . The field equations then reduce to

$$H^2 = \frac{1}{3f'} \left[\frac{Rf' - f}{2} - 3H\dot{R}f'' \right], \quad (2.2)$$

$$2\dot{H} + 3H^2 = -\frac{1}{f'} \left[f'''(\dot{R})^2 + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf') \right], \quad (2.3)$$

where an overdot denotes differentiation with respect to t . We assume that $f' > 0$ to have a positive effective gravitational coupling; furthermore, modified gravity theories suffer from violent instabilities unless $f'' \geq 0$ ([20, 21, 8, 9] — see also [22]) and it seems appropriate to exclude the case $f'' < 0$ on a physical basis. The field equations are of fourth order in $a(t)$. However, when the curvature index $k = 0$, a only appears in the combination $H \equiv \dot{a}/a$. Since the Hubble parameter H is a cosmological observable, it is convenient to adopt it as the dynamical variable; then, the field equations (2.2) and (2.3)

are of third order in H . The elimination of a is not possible when the curvature index $k \neq 0$, or when a fluid with density $\rho = \rho(a)$ is included in the picture. We can describe the dynamics in the three-dimensional phase space (H, R, \dot{R}) , then the Hamiltonian constraint (2.2) implies that the orbits of the solutions lie on an energy surface Σ and there is an expression for \dot{R} for any given value of the other two variables (H, R) . This is given explicitly by eq. (2.2) as

$$\dot{R}(H, R) = \frac{Rf' - f - 6f'H^2}{6Hf''}, \quad (2.4)$$

where we assume that $f'' > 0$. It is therefore possible to eliminate the variable \dot{R} given a pair (H, R) . Note that there is only one value of \dot{R} for each given value of (H, R) . This situation is different from a general scalar-tensor theory in which one obtains *two* values $\dot{\phi}_{\pm}(H, \phi)$ for any value of the pair (H, ϕ) (ϕ being the scalar field of gravitational origin) because the Hamiltonian constraint is a quadratic equation in $\dot{\phi}$ in which the term $\dot{\phi}^2$ is multiplied by the Brans-Dicke “parameter” $\omega(\phi)$ (which in general scalar-tensor gravities becomes a function of ϕ instead of the constant parameter appearing in Brans-Dicke theory [23, 24]). The scalar-tensor equivalent of $f(R)$ gravity has $\omega = 0$ and no quadratic terms in $\dot{\phi}$ — see the next section. This situation reflects the fact that $\dot{R}(H, R)$ is single-valued.

By eliminating the variable \dot{R} , the orbits of the solutions of eqs. (2.2) and (2.3) are confined to the two-dimensional, curved, surface Σ described by eq. (2.4) in the three-dimensional space (H, R, \dot{R}) . By contrast, in general scalar-tensor theories with $\omega \neq 0$, $\dot{\phi} = \dot{\phi}_{\pm}(H, \phi)$ is a double-valued function and the corresponding surface Σ_{\pm} is two-sheeted [24]. As a consequence of the two-dimensional nature of Σ , there can be no chaos in the dynamics. Although the standard Poincaré-Bendixson theorem [25] only applies to a flat, compact and simply connected two-dimensional phase space, it is rather straightforward to prove the absence of chaos in the two-dimensional, two-sheeted phase space of scalar-tensor gravity [26], and the proof is easily extended to the simpler one-sheeted phase space of $f(R)$ gravity.

As a consequence of the fact that the phase space is constituted of a single sheet, the projections of the orbits onto the (H, R) plane can not intersect one another, contrary to what happens in the general scalar-tensor case [24]. In general, there is a *forbidden region* in the phase space, corresponding to $H^2 < 0$ and here called $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 \equiv \left\{ (H, R) : Rf'(R) - f(R) - 6H\dot{R}(H, R)f''(R) < 0 \right\} \quad (2.5)$$

and

$$\mathcal{F}_2 \equiv \{ (H, R) : f(R), f'(R), \text{ or } f''(R) \text{ are not defined} \}. \quad (2.6)$$

In addition, we assume that $f' > 0$ to ensure a positive gravitational coupling and that $f'' > 0$ to avoid instabilities, which restricts further the region dynamically allowed to the orbits of the solutions. The boundary of the connected subset \mathcal{F}_1 of the forbidden region is

$$\mathcal{B} \equiv \{ (H, R, \dot{R}) : f'(R)H^2 = 0 \}, \quad (2.7)$$

which defines a curve in the surface Σ , along which the effective energy density ρ_{eff} given by eq. (2.14) below vanishes.

A difference with respect to general scalar-tensor cosmology stands out: in these theories, the quadratic equation for the time derivative $\dot{\phi}$ of the Brans-Dicke-like scalar $\phi(t)$ provides two distinct roots $\dot{\phi}_{\pm}$ which differ by a square root term entering with positive or negative sign, and there is a forbidden region corresponding to the argument of this square root being negative [24]. As a result, the forbidden region intersects the two-sheeted energy surface Σ , creating some “holes” in this surface. The orbits of the solutions can change sheet at the boundary of these holes. In $f(R)$ cosmology instead, there is no such square root term and the forbidden region does not intersect the energy surface Σ .

Having chosen (H, R) as variables, the equilibrium points are given by $(\dot{H}, \dot{R}) = (0, 0)$ and are necessarily the de Sitter spaces

$$(H, R, \dot{R}) = (H_0, 12H_0^2, 0) = \left(\pm \sqrt{\frac{f_0}{6f'_0}}, \frac{2f_0}{f'_0}, 0 \right), \quad (2.8)$$

where $f_0 \equiv f(R_0)$, $f'_0 \equiv f'(R_0)$, *etc.* These fixed points only exist when the condition $f_0/f'_0 \geq 0$ is satisfied. There is only one condition for the existence of the de Sitter space, contrary to general scalar tensor gravity [24] because eq. (2.2) is not independent of eq. (2.3) and there is only one independent equation left. According to the observational data available [1], the present state of the universe is characterized by an effective equation of state $P \simeq -\rho$, i.e., if the $f(R)$ model applies, we are near a de Sitter fixed point.

If $f_0 \geq 0$ and $f'_0 > 0$, two de Sitter equilibrium points (one contracting and one expanding) always exist in the $\dot{R} = 0$ plane with $H_0^{(\pm)} = \pm \sqrt{\frac{f_0}{6f'_0}}$; they degenerate into a single point, a Minkowski space, if $f(R_0) = 0$, in which case they lie on the boundary \mathcal{B} of the forbidden region \mathcal{F}_1 . The latter approaches the energy surface $\dot{R} = \dot{R}(H, R)$ at these Minkowski space points, which are the only possible global Minkowski solutions.

The stability of the de Sitter equilibrium points of modified gravity with respect to inhomogeneous (space and time-dependent) perturbations was studied in Refs. [27] by using the covariant and gauge-invariant formalism of Bardeen-Ellis-Bruni-Hwang [28], in Hwang’s version valid for generalized gravity [29]. The result is the covariant and gauge-invariant condition for linear stability of de Sitter space with respect to inhomogeneous perturbations

$$\frac{(f'_0)^2 - 2f_0f''_0}{f'_0f''_0} \geq 0. \quad (2.9)$$

The contracting de Sitter spaces with $H_0 < 0$ are always unstable [27]. To linear order, expanding de Sitter spaces are always stable with respect to tensor modes[‡]. The equilibrium points $(H, R, \dot{R}) = (H_0, 12H_0^2, 0)$ always lie in the (H, R) plane of the phase space.

[‡] It turns out that (2.9) coincides with the stability condition of de Sitter space with respect to *homogeneous* perturbations [30, 31].

Examples of the phase space geometry for two choices of the function $f(R)$ in the gravitational action (1.1) are presented in fig. 1 and fig. 2.

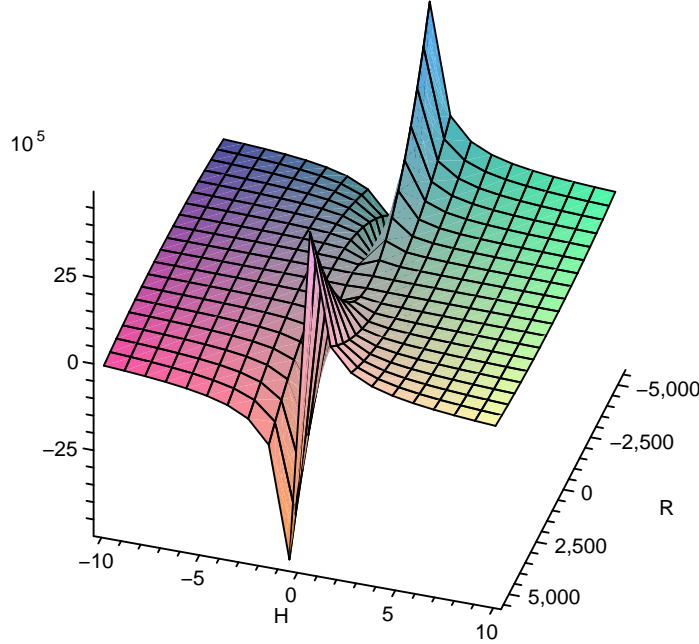


Figure 1. Phase space portrait for the model $f(R) = R + \alpha R^2$, with $\alpha = 0.0001$. The vertical axis shows \dot{R} and R, H, \dot{R}, α are measured in arbitrary units.

It is useful to obtain the effective Lagrangian L and Hamiltonian E for $f(R)$ cosmology in FLRW space with a and R as Lagrangian coordinates. To the best of our knowledge, the effective Lagrangian and Hamiltonian (2.10) and (2.11) below have not been presented in the literature, and will be used in future work. For example, they can be used in the search for point-like symmetries (see Ref. [32]). An analysis of the field equations leads one to conclude that the effective Lagrangian is

$$L(a, R, \dot{a}, \dot{R}) = a^3 \left[6H^2 f' + 6H f'' \dot{R} + f' R - f \right] \quad (2.10)$$

and that the effective energy[§] is

$$E(a, R, \dot{a}, \dot{R}) = a^3 \left(6H^2 f' + 6H f'' \dot{R} - f' R + f \right). \quad (2.11)$$

The canonical momenta conjugated to the generalized coordinates a and R are

$$p_a = \frac{\partial L}{\partial \dot{a}} = 6a^2 \left(2H f' + f'' \dot{R} \right), \quad p_R = \frac{\partial L}{\partial \dot{R}} = 6a^3 H f''. \quad (2.12)$$

The Hamiltonian constraint (2.2) corresponds to $E = 0$. By choosing a and R as canonical variables, the Euler-Lagrange equation $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{R}} \right) - \frac{\partial L}{\partial R} = 0$ gives the well

[§] The quantity E can also be regarded as an effective Hamiltonian but it does not contain explicitly the generalized momenta.

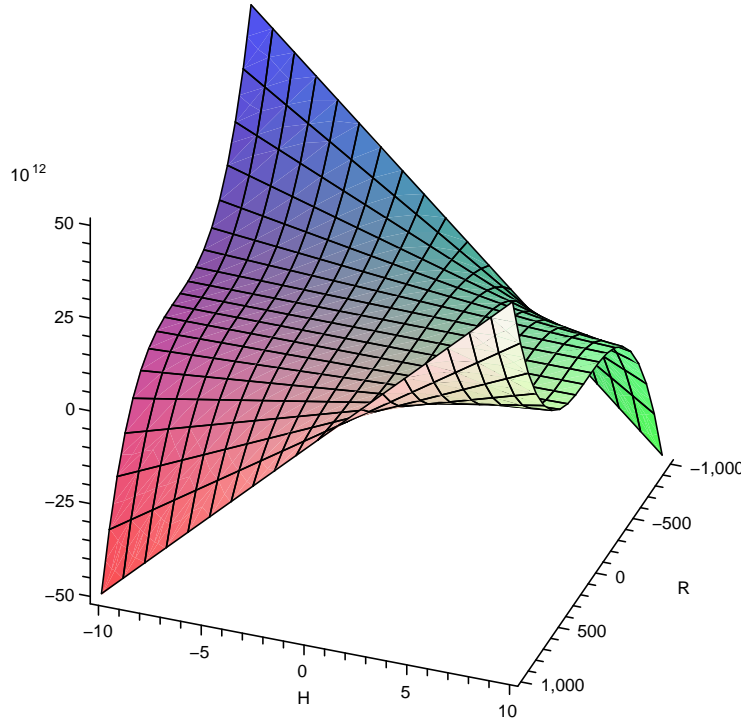


Figure 2. Phase space for the model $f(R) = R - \frac{\mu^4}{R}$, with $\mu^4 = 0.0001$, in arbitrary units, with \dot{R} on the vertical axis.

known relation $R = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)$, while the second equation $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{a}} \right) - \frac{\partial L}{\partial a} = 0$ yields eq. (2.3). For non-spatially flat FLRW universes ($k \neq 0$), the Hamiltonian constraint takes the form

$$H^2 = \frac{1}{3f'} \left[\frac{Rf' - f}{2} - 3H\dot{R}f'' \right] - \frac{k}{a^2}, \quad (2.13)$$

in which the scale factor a appears explicitly, and one can not use H as variable instead of a . The surface Σ studied for $k = 0$ separates the region of the phase space accessible to the orbits of the solutions for $k > 0$ from the region corresponding to $k < 0$. The orbits can not cross the surface Σ and, in the case $k = 0$, instead, they lie completely in Σ . A similar situation was recognized in an early study of inflation with a massive scalar field in the context of Einstein gravity [33].

2.1. The effective equation of state

Using eqs. (2.2) and (2.3) and identifying the effective gravitational coupling with $1/f'(R)$, one can write the effective energy density and pressure for $f(R)$ models as follows:

$$\rho_{eff} = \frac{Rf' - f}{2} - 3H\dot{R}f'', \quad (2.14)$$

$$P_{eff} = \dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf') . \quad (2.15)$$

The effective energy density ρ_{eff} is non-negative, as can be seen from inspection of eq. (2.2). The effective equation of state parameter w_{eff} can be expressed as

$$w_{eff} \equiv \frac{P_{eff}}{\rho_{eff}} = \frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')}{\frac{Rf' - f}{2} - 3H\dot{R}f''} . \quad (2.16)$$

Equation (2.2) guarantees that the denominator on the right hand side of eq. (2.16) is strictly positive, hence, the sign of the effective equation of state is determined by the numerator. Type Ia supernovae yield an effective equation of state $w \approx -1$ at present. For the model to mimic the de Sitter equation of state with $w_{eff} = -1$, it must be

$$\frac{f'''}{f''} = \frac{\dot{R}H - \ddot{R}}{(\dot{R})^2} . \quad (2.17)$$

It is convenient to introduce the quantity $\psi(R) \equiv f'(R)$ to write

$$w_{eff} = -1 + 2 \frac{(\ddot{\psi} - H\dot{\psi})}{R\psi - f - 6H\dot{\psi}} = -1 + \frac{(\ddot{\psi} - H\dot{\psi})}{3\psi H^2} . \quad (2.18)$$

Alternatively, the deviation from the de Sitter equation of state $w = -1$ can be parametrized by

$$\rho_{eff} + P_{eff} = \frac{\ddot{\psi} - H\dot{\psi}}{\psi} = \frac{\dot{\psi}}{\psi} \frac{d}{dt} \left[\ln \left(\frac{\dot{\psi}}{a} \right) \right] . \quad (2.19)$$

According to eq. (2.19), an exact de Sitter solution corresponds to $\dot{\psi} = f''(R)\dot{R} = 0$, or to $\dot{\psi} = Ca(t) = Ca_0 e^{H_0 t}$, where $C \neq 0$ is an integration constant. It is easy to see that the second solution for $\psi(t)$ is not acceptable because it leads to the absurd equation $f''(R)\dot{R} = Ca_0 e^{H_0 t}$ in which the left hand side is time-independent (for a de Sitter solution) while the right hand side is not.

3. The equivalent scalar-tensor theory

A modified gravitational action of the form (1.1) can be recast into the form of an equivalent scalar-tensor theory ([34, 35]; see [16] for a mapping into general relativity with a minimally coupled scalar). When $f'' \neq 0$, the modified gravity action (1.1) can be rewritten as the scalar-tensor theory

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [\psi(\phi)R - V(\phi)] + S^{(matter)} , \quad (3.1)$$

where

$$\psi(\phi) = f'(\phi) , \quad V(\phi) = \phi f' - f , \quad (3.2)$$

and the Brans-Dicke parameter is $\omega = 0$ [34, 35]. The Ricci curvature scalar R is identified with the scalar field degree of freedom ϕ and the mapping to scalar-tensor

variables can be seen as a Legendre transformation (see Ref. [17] for a discussion, and Ref. [36] about the physical equivalence). In fact, variation of the action (3.1) with respect to ϕ yields

$$R \frac{d\psi}{d\phi} - \frac{dV}{d\phi} = 0 , \quad (3.3)$$

which in turn yields $\phi = R$ if $f'' \neq 0$. There is no kinetic term for the scalar ϕ in the action (3.1), but this quantity is dynamical because $R = \phi$ obeys the dynamical equation (1.2).

In the metric (2.1) the field equations become

$$H^2 = \frac{1}{6\psi} \left[V(\phi) - 6H\psi'\dot{\phi} \right] , \quad (3.4)$$

$$\dot{H} = \frac{1}{2\psi} \left(H\psi'\dot{\phi} - \psi'\ddot{\phi} - \psi''\dot{\phi}^2 \right) , \quad (3.5)$$

and (3.3), or

$$H^2 = \frac{1}{6f'} \left(\phi f' - f - 6Hf'''\dot{\phi} \right) , \quad (3.6)$$

$$\dot{H} = \frac{1}{2f'} \left(Hf''\dot{\phi} - f''\ddot{\phi} - f'''\dot{\phi}^2 \right) , \quad (3.7)$$

$$(R - \phi) f'' = 0 . \quad (3.8)$$

By viewing the Hamiltonian constraint (3.4) as an algebraic equation for $\dot{\phi}$, one can eliminate $\dot{\phi}$ from the three-dimensional phase space $(H, \phi, \dot{\phi})$ (which corresponds to the phase space (H, R, \dot{R}) of Sec. 2), obtaining

$$\dot{\phi}(H, \phi) = \frac{1}{6Hf''} (\phi f' - f - 6f'H^2) . \quad (3.9)$$

This represents a one-sheeted energy surface, which corresponds to the special case $\omega = 0$ of scalar-tensor cosmology [24], as remarked in the previous section. The equilibrium points $(\dot{H}, \dot{\phi}) = (0, 0)$ are de Sitter spaces with constant scalar field and satisfy

$$(H_0, \phi_0) = \left(\pm \sqrt{\frac{f_0}{6f'_0}}, \frac{2f_0}{f'_0} \right) ; \quad (3.10)$$

they lie in the $\dot{\phi} = 0$ plane. In general scalar-tensor gravity it is possible to have de Sitter solutions with non-constant scalar field (which, however, are not fixed points). In the scalar-tensor equivalent of $f(R)$ gravity this is not possible because of the constraint $\phi = R = 12H_0^2$ which forces ϕ to be constant for all de Sitter solutions, which have constant Hubble parameter H_0 .

The gauge invariant stability condition for these de Sitter equilibrium points with respect to inhomogeneous perturbations, obtained in Ref. [30], is again

$$\frac{(f'_0)^2 - 2f_0f''_0}{f'_0f''_0} \geq 0 . \quad (3.11)$$

The effective Lagrangian and Hamiltonian for the equivalent scalar-tensor theory (3.1) are now given by

$$L(a, \phi, \dot{a}, \dot{\phi}) = a^3 (6aH^2\psi + 6H\dot{\psi} + V) \quad (3.12)$$

and

$$E(a, \phi, \dot{a}, \dot{\phi}) = a^3 (6H^2\psi + 6H\dot{\psi} - V) . \quad (3.13)$$

These can be obtained from well known Lagrangians of scalar-tensor theories ([37] and references therein) and do not constitute an original result, contrary to the L and E given by eqs. (2.10) and (2.11).

4. Discussion and conclusions

The phase space view of a spatially homogeneous and isotropic universe is useful for building models of the current acceleration of our cosmos and in trying to discover what fuels it. The high degree of symmetry of the FLRW metric (2.1) considered turns the Einstein equations into ordinary differential equations, for which the Cauchy problem is trivial and is covered by well-known theorems. Instead, the initial value problem of $f(R)$ gravity for general metrics has not been discussed, except for specific choices of the function $f(R)$ [38, 34], and will be addressed elsewhere.

The dynamical system approach is a powerful tool which is appropriate to study the dynamics for general initial conditions when exact solutions can not be obtained (i.e., in most situations). As such, dynamical systems theory is widely applied in cosmology (see, e.g., Refs. [39, 40]). The theory of dynamical systems has been applied repeatedly also to $f(R)$ cosmology, but usually for special choices of the function $f(R)$ [41]. Our goal is to study the phase space of $f(R)$ cosmology without committing to a specific choice of the function $f(R)$. Motivated by the recent observations of type Ia supernovae [1] and by the current cosmic microwave background experiments [2], we have restricted the scope of our analysis to spatially flat FLRW cosmologies. Advantages of our approach include its generality, the use of physical observables as dynamical variables, and the ease of comparison with the general picture of the phase space for the equivalent scalar-tensor gravity. The obvious limitation is that the phase space portrait is necessarily incomplete without choosing a specific form of $f(R)$. However, in the absence of compelling physical indications on the form of this function, it is hoped that our results on the geometry of the phase space, the equilibrium points and their stability, and the effective Lagrangians and Hamiltonians can be used as a preliminary step and a guide to further understanding the cosmological dynamics in any specific $f(R)$ theory. The extension to more general gravity theories of the form $f(R, R_{ab}R^{ab}, R_{abcd}R^{abcd})$ will be discussed elsewhere.

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